POINTWISE ASYMPTOTICS FOR THE JUMPS OF ERGODIC AVERAGES

P. J. CATUOGNO AND S. E. FERRANDO

Abstract. We study the pointwise asymptotic behaviour for the number of jumps of ergodic averages as the size of the oscillations decreases to zero. The study is carried out in the setting of Chacon-Ornstein averages. We find that under rather general conditions there exists a pointwise almost uniform asymptotics that relates the number and size of the jumps. The proof makes use of Bishop's upcrossing inequalities.

1. Introduction

It is well known that a sequence of ergodic averages can exhibit any speed of convergence ([8]). This multiplicity of rates of convergence does not exclude the possibility to obtain useful information on *spatial* characteristics (e.g. oscillations) of ergodic averages. This information could be used, for example, to monitor convergence of a sequence of ergodic averages. In this paper we describe a result on *majorizing pointwise asymptotics*; results for *minorizing asymptotics* are also possible and are analogous to the topic of reverse inequalities ([6]). These results will be reported in another publication.

We first comment on the main result of the paper, precise definitions of all the quantities involved are given elsewhere in the paper. Let $J_\eta(x)$ be the maximum number of η-jumps (or oscillations) for an ergodic average at point x, the asymptotics we are interested is when $\eta \searrow 0$. Under appropiate conditions we prove that if λ is an increasing function so that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda(2^n)} < \infty, \tag{1}$$

we have the following asymptotic result when $\eta \searrow 0$:

$$J_\eta(x)\ \eta^2 = o\left(\lambda(1/\eta)\right) \text{ a.u.} \tag{2}$$

where the "o (.) a.u." notation denotes an almost uniform convergence in x.

In particular, for any given real number $\epsilon > 0$ and integer $r > 0$ the above results implies:

$$J_\eta(x)\ \eta^2 = o\left((\log_1(1/\eta)) \dots (\log_{r-1}(1/\eta))(\log_r(1/\eta))^{1+\epsilon}\right) \text{ a.u.} \tag{3}$$

where we use the notation $\log_r(x) = \log \circ \dots \circ \log(x)$, i.e. we have composed the $\log(.)$ function r times.

Date: May 17, 2001.

1991 *Mathematics Subject Classification.* Primary:28D99 ; secondary:47A35.

Key words and phrases. Cesaro averages. Upcrossing inequalities. Pointwise a.e. convergence. Asymptotics.

Research partially supported by NSERC.

The proof of (2) relies mainly on Bishop's upcrossing inequalities to establish an integral inequality for the jumps. This is done in a geometric way by relating upcrossings and jumps. Then, we use an argument similar to the one of Gal and Koksma ([4], see also [1]) that allows to deduce pointwise asymptotics from integral bounds. We work in the general setting introduced by Bishop in [2], this allows us to obtain results for the Chacon-Ornstein averages as well as for related sequences. Results on Cesaro averages follow as a special case.

For the case of Cesaro averages, as was observed by an anonymous referee, our main result can be proved using a known weak $(2,2)$ estimative for the jumps (this inequality is a consequence of Lemma 2 and Chebyshev inequality) and some of the arguments in our paper. Actually, these techniques will give results for the setting of L^p spaces with $1 \leq p$. As a comparison, notice that Theorem 2 for the case of Cesaro averages is essentially an L^2 result. Presently, our techniques do not give the L^p results, Bishop's methods need to be refined to make them capable to extend the techniques from our paper to other L^p spaces. On the other hand, our techniques work for the setting of Chacon-Ornstein where the approach through weak inequalities is not presently available. More generally, our approach can be used in problems where only upcrossing inequalities are available and jump inequalities are not known.

The paper is organized as follows. In Section 2 we introduce the basic definitions and background results then we proceed to prove the main result of the paper, Theorem 2. In Appendix A we prove results which are needed to obtain an *explicit* modulus of almost uniform convergence. In Appendix B we state Bishop's general result on upcrossing inequalities and specialize that result to the two cases treated in our paper.

2. Pointwise Asymptotics for Jumps

We use the following notation and assumptions: $(X, \mathcal{F}, \mu)$ is an arbitrary measure space. Functions are assumed to be real valued and equalities and inequalities of functions are meant in the almost everywhere sense. For the most part we work with a positive linear contraction T from L^1 to L^1. Given a sequence of functions $g_n(x)$ we will refer to the whole collection as a single object by means of $g = \{g_n(x)\}$. An *admissible* sequence $q = \{q_k(x)\}$ is a collection of measurable functions that satisfy $q_k(x) \geq 0$ and $Tq_k \leq q_{k+1}$, we also assume $q_0 > 0$ a.e. Let $f(x)$ denote an arbitrary mesurable function for the moment; some of the results will be stated using the abstract notation $S_n f(x)$, $n = -1, 0, 2, \ldots$ ($S_{-1} f(x) = 0$) which will stand for any of the following two sequences:

i) The Chacon-Ornstein averages $S_n f(x) = A_n f(x) = \frac{\sum_{k=0}^n T^k f(x)}{\sum_{k=0}^n q_k(x)}$. These averages specialize to Cesaro averages $A_n f(x) = \sum_{k=0}^n T^k f(x)/(n+1)$ if $q_k(x) = 1$ and $T1 \leq 1$.

ii) Powers of T, namely $S_n f(x) = P_n f(x) = \frac{T^n f(x)}{\sum_{k=0}^n q_k(x)}$. This sequence is usually studied in connection with the Chacon-Ornstein averages.

Next we make precise the meaning of the symbol "little o" in our paper.

Definition 1. Almost Uniform Asymptotics:
Given a sequence of measurable functions $a_n(x)$ and a sequence $c(n)$ the notation $a_n(x) = o(c(n))$ a.u. (or $\lim_{n\to\infty} (a_n(x)/c(n)) = 0$ a.u.) means the following: $\forall\ \epsilon >$

0 there exists a measurable set $A \subset X$ with $\mu(A^c) < \epsilon$ and such that the sequence $\frac{a_n(x)}{c(n)}$ converges uniformly to 0 on A as n approaches ∞.

Remark 1. *By Egorov's theorem almost uniform (a.u.) convergence follows from almost everywhere convergence for finite measure spaces. Our point of emphasizing a.u. convergence is that our results will hold in arbitrary measure spaces and that we give explicit modulus of convergence.*

Definition 2. Jumps:
Given a sequence of functions $g = \{g_n(x)\}$, $n = -1, 0, \dots$ ($g_{-1}(x) = 0$), a fixed integer $K > 0$, a real number $\eta > 0$ and $x \in X$, define

$$J_{\eta,K}(g,x) = \sup\{k : \xi = (t_r)_{r=0,\dots,k}\}$$

where ξ satisfies:

$$-1 \leq t_0 < t_1 < t_2 < \dots < t_k \leq K$$

and

$$|g_{t_{r+1}}(x) - g_{t_r}(x)| \geq \eta, \quad \textit{for all } r = 0, \dots, k-1. \tag{4}$$

Also define

$$J_\eta(g,x) = \sup\{J_{\eta,K}(g,x) : K > 0\}$$

the function $J_\eta(g)$ will be referred to as the number of η-jumps for the given sequence g. Results involving $J_\eta(g,x)$ when $g_n(x)$ equal the Cesaro averages are developed in [5]. In [7] it is proven, among many other results, that the function J_η is not integrable if $f \in L^1$.

Definition 3. Upcrossings:
Given a sequence of functions $g = \{g_n(x)\}$, $n = -1, 0, \dots$ ($g_{-1}(x) = 0$), an integer $K > 0$, real numbers α, $\eta > 0$ and $x \in X$ define

$$U_{\eta,K,\alpha}(g,x) = \sup\{k : \zeta = (u_r, v_r)_{r=1,\dots,k.}\} \tag{5}$$

where the sequence ζ satisfies,

$$-1 \leq u_1 < v_1 < u_2 < \dots < v_k \leq K \tag{6}$$

$$g_{u_r}(x) \leq \alpha \quad \textit{and} \quad g_{v_r}(x) \geq (\alpha + \eta) \tag{7}$$

for $r = 1, \dots, k$. The function $U_{\eta,K,\alpha}(g,x)$ will be referred to as the number of upcrossings through the interval $[\alpha, \alpha + \eta]$ (see [2]) for the given sequence g.

The following theorem allows us to derive the pointwise asymptotics in Theorem 2. If one only wants to establish only a.e. convergence the argument is simpler.

Theorem 1. *Given a sequence of nonnegative functions $f_i \in L^1(X)$, $i = 0, 1, \dots$, a function ϕ defined on the positive integers such that*

$$\sum_{i=0}^{n-1} \int f_i(x)\, d\mu(x) = O(\phi(n)) \tag{8}$$

and φ a positive, increasing function defined on the positive integers so that for some sequence $k_1 < k_2 < \dots$ satisfies

$$\sum_{n=0}^{\infty} \frac{\phi(k_{n+1})}{\varphi(k_n)} < \infty \tag{9}$$

then

$$\lim_{N\to\infty} \frac{1}{\varphi(N)} \sum_{i=0}^{N-1} f_i(x) = 0 \ a.u. \tag{10}$$

Proof. We define

$$a_n(x) = \frac{1}{\varphi(k_n)} \sum_{i=0}^{k_{n+1}-1} f_i(x)$$

From (8) and (9) it follows that

$$\sum_{n=0}^{\infty} \int a_n(x)\ d\mu(x) \leq C \sum_{n=0}^{\infty} \frac{\phi(k_{n+1})}{\varphi(k_n)} < \infty. \tag{11}$$

We will use now the proof of Lemma 3 and its notation. Let $a_n = \frac{C\ \phi(k_{n+1})}{\varphi(k_n)}$, $n = 0, 1, \ldots$, with associated sequence q_n. Given $\delta_1 > 0$ and $\delta_2 > 0$, there exists a measurable set B such that $\mu(B) \leq \delta_1$ and an integer N_2 such that:

$$a_n(x) \leq \delta_2 \ \forall\ x \in B^c \text{ and } \forall\ n \ \geq N_2. \tag{12}$$

Take $P_2 = k_{N_2}$, to establish (10) it is enough to prove that:

$$\frac{1}{\varphi(P)} \sum_{i=0}^{P-1} f_i(x) \leq \delta_2,\ \forall\ x \in\ B^c \text{ and } \forall\ P\ \geq\ P_2. \tag{13}$$

Let $P \geq P_2$ and let n_P be the unique integer such that $k_{n_P} \leq P < k_{n_P+1}$, then $N_2 \leq n_P$ and from equation (12), the fact that φ is an increasing function on the integers and $f_i(x) \geq 0$ we obtain (13). □

Definition 4. We say that a given sequence $g = \{g_n(x)\}$ crosses the interval $[\alpha, \alpha + \eta]$ from left to right if there are integers $-1 \leq n_1 < n_2$ satisfying $g_{n_1}(x) \leq \alpha$ and $g_{n_2}(x) \geq \alpha + \eta$. Similarly for a crossing from right to left. Finally, we say that $g_n(x)$ crosses the interval $[\alpha, \alpha + \eta]$ if it crosses the interval from left to right or from right to left.

Definition 5. Set $\alpha_i = i\ \frac{\eta}{2}$ for $i = 0, 1, \ldots$ and define $U_{\eta/2,K}(g, x) = \sum_{i=0}^{\infty} U_{\eta/2,K,\alpha_i}(g, x)$.

The following simple (but crucial) lemma states that if the sequence is bounded from below, we can add the upcrossings to bound the number of jumps. We only need this result for the case when the sequence g is nonnegative.

Lemma 1. *Let $g = \{g_n(x)\}$ and $g_n(x) \geq 0$ then*

$$J_{\eta,K}(g, x) \leq 2\ U_{\frac{\eta}{2},K}(g, x). \tag{14}$$

Proof. Fix x once and for all. We will prove by induction in $r \geq 0$ that given integers $-1 = t_0 < t_1 < \ldots < t_r \leq K$ such that

$$|g_{t_{k+1}}(x) - g_{t_k}(x)| \geq \eta \text{ for all } k = 0, \ldots, r-1 \tag{15}$$

and if we let $k_i(t_r)$ denote the number of times $g_{t_k}(x), k = 0, \ldots, r$, crosses the interval $[\alpha_i, \alpha_i + \eta/2]$, then:

if $k_i(t_r)$ is odd

$$k_i(t_r) < 2\ U_{\eta/2,t_r,\alpha_i}(g, x) \tag{16}$$

and if $k_i(t_r)$ is even then

$$k_i(t_r) \leq 2\ U_{\eta/2,t_r,\alpha_i}(g,x) \tag{17}$$

For simplicity set $k_i(-1) = 0$. Given integers $-1 = t_0 < t_1 < \ldots < t_{r+1} \leq K$ satisfying (15) we need to establish (16) and (17) for $k_i(t_{r+1})$. It is enough to consider only the intervals $[\alpha_i, \alpha_i + \eta/2]$ for which $k_i(t_{r+1}) = k_i(t_r) + 1$. We break the analysis in two cases:
Case I: $k_i(t_r)$ is odd, then it follows that $k_i(t_{r+1}) = k_i(t_r) + 1 \leq 2\ U_{\eta/2,t_r,\alpha_i}(g,x) \leq 2\ U_{\eta/2,t_{r+1},\alpha_i}(g,x)$ hence (17) holds for the case when $k_i(t_{r+1})$ is even.
Case II: $k_i(t_r)$ is even, then it follows from our definitions that

$$g_{t_r}(x) \leq \alpha_i \text{ and } g_{t_{r+1}}(x) \geq (\alpha_i + \eta/2). \tag{18}$$

From the inductive hypothesis we know that $k_i(t_r) \leq 2\ U_{\eta/2,t_r,\alpha_i}(g,x)$.
It follows from the definition of $k_i(t_{r+1})$ and the fact that it is an odd number that $U_{\eta/2,t_{r+1},\alpha_i}(g,x) \geq \frac{(k_i(t_{r+1})+1)}{2}$, hence $k_i(t_{r+1}) < 2\ U_{\eta/2,t_{r+1},\alpha_i}(g,x)$ as was required to prove. Let now $J_{\eta,K}(g,x) = n$, therefore, there is an increasing sequence of integers $u_i = u_i(x)$, $-1 \leq u_0 < u_1 < \ldots, < u_n \leq K, i = 0, \ldots, n$ such that $|g_{u_{k+1}}(x) - g_{u_k}(x)| \geq \eta$ for all $k = 0, \ldots, n$ holds. Given that g is bounded below by zero, we can take $u_0 = -1$ without loss of generality. We then have

$$n \leq \sum_{i=0}^{\infty} k_i(u_n) \leq 2\ \sum_{i=0}^{\infty} U_{\eta/2,K,\alpha_i}(g,x) = 2\ U_{\frac{\eta}{2},K}(g,x) \tag{19}$$

where the first inequality follows from the definitions and the second one follows from equations (16) and (17). □

For the definition and a discussion of the function $w_{\eta,K,\alpha}(S,x)$ which appears in the proof of the next lemma we refer to Appendix B, at this point we just notice that $U_{\eta,K,\alpha}(S,x) \leq w_{\eta,K,\alpha}(S,x)$ (where the notation $S = \{S_n(x)\}$ was introduced in Section 1).

Lemma 2. *Let $q = \{q_n(x)\}$ be an admissible sequence, assume both f and $\frac{f^2}{q_0}$ belong to $L^1(X)$ and $f(x) \geq 0$. Then:*

$$\int q_0(x)\ J_{\eta,K}(S,x)\ d\mu(x) \leq \frac{4}{\eta^2} \int \frac{f^2(x)}{q_0(x)}\ d\mu + \frac{6}{\eta} \int f(x)\ d\mu. \tag{20}$$

Proof. Similarly to Definition 5, let $\alpha_i = i\ \frac{\eta}{2}$ for $i = 0, \ldots$ and set:

$$w_{\frac{\eta}{2},K}(S,x) = \sum_{i=0}^{\infty} w_{\frac{\eta}{2},K,\alpha_i}(S,x).$$

From Lemma 1 applied to the sequence $S = \{S_n(x)\}$ we obtain:

$$J_{\eta,K}(S,x) \leq 2\ w_{\frac{\eta}{2},K}(S,x). \tag{21}$$

Notice that if $N(x) = \lfloor \frac{2\ f(x)}{\eta q_0(x)} \rfloor$ is the integer part of $\frac{2\ f(x)}{\eta q_0(x)}$ it follows that

$$\sum_{i=0}^{i \leq N(x)} (f(x) - \frac{i\ \eta\ q_0(x)}{2}) \leq \left(\frac{f^2(x)}{\eta q_0(x)} + \frac{3\ f(x)}{2} \right). \tag{22}$$

We remark that in Appendix B we indicate that Theorem 3 is applicable to both sequences $S_n(x) = A_n f(x)$ and $S_n(x) = P_n f(x)$. Using (22) and Theorem 3 we compute:

$$\begin{aligned}\sum_{i=0}^{\infty} \int q_0(x)\ w_{\frac{\eta}{2},K,\alpha_i}(S,x)\ d\mu(x) &\leq \frac{2}{\eta} \sum_{i=0}^{\infty} \int (f(x) - \alpha_i\ q_0(x))_+\ \ d\mu(x) \\ &\leq \frac{2}{\eta} \sum_{i=0}^{\infty} \int_{\{f>\alpha_i q_0\}} (f(x) - \alpha_i\ q_0(x))\ \ d\mu(x) \\ &\leq \frac{2}{\eta} \int \sum_{i=0}^{i\leq N(x)} (f(x) - \frac{i\ \eta\ q_0(x)}{2})\ d\mu(x) \\ &\leq \frac{2}{\eta^2} \int \frac{f^2(x)}{q_0(x)}\ d\mu(x) + \frac{3}{\eta} \int f(x)\ d\mu(x).\end{aligned}$$

Hence by Fubini's theorem,

$$\begin{aligned}\int q_0(x)\ J_{\eta,K}(S,x)\ d\mu(x) &\leq \int 2\ q_0(x)\ w_{\frac{\eta}{2},K}(S,x)\ d\mu(x) \\ &\leq \frac{4}{\eta^2} \int \frac{f^2(x)}{q_0(x)}\ d\mu + \frac{6}{\eta} \int f(x)\ d\mu.\end{aligned}$$

Hence (20) is proven. $\square$

Theorem 2. *Let $q = \{q_n(x)\}$ be an admissible sequence, assume both f and $\frac{f^2}{q_0}$ belong to $L^1(X)$ and ζ is a positive and increasing function defined on the integers so that*

$$\sum_{n=1}^{\infty} \frac{1}{\zeta(2^n)} < \infty. \tag{23}$$

Then for any nonincreasing sequence η_k tending to 0, we have the following asymptotic result:

$$J_{\eta_{2N}}(S,x)\ \eta_N^2 = o\left(\zeta(2N)\right)\ \ a.u. \tag{24}$$

Proof. Notice that to prove (24) it is enough to consider the case when $f(x) \geq 0$, this is so because if $f = f_+ - f_-$ we have $J_\eta(f)(S,x) \leq J_{\eta/2}(f_+)(S,x) + J_{\eta/2}(f_-)(S,x)$. Let η_k be a nonincreasing sequence such that $\eta_k \searrow 0$.
Using (20) we have

(25)
$$\sum_{k=0}^{N-1} \int q_0(x)\ \eta_k^2 J_{\eta_k}(S,x)\ d\mu(x) \leq 4\ N \int \frac{f^2(x)}{q_0(x)}\ d\mu(x) + 6\ \eta_0\ N\ \int f(x)\ d\mu(x).$$

If we take $\varphi(N) = N\ \zeta(N)$, $\phi(N) = N$ and $k_n = 2^{n+1}$ we have

$$\sum_{n=0}^{\infty} \frac{\phi(k_{n+1})}{\varphi(k_n)} = \sum_{n=1}^{\infty} \frac{2}{\zeta(2^n)} < \infty. \tag{26}$$

If we let $f_i(x) = \eta_i^2\ q_0(x)\ J_{\eta_i}(S,x)$ in Theorem 1, equations (8) and (9) from Theorem 1 hold and (10) implies

$$\sum_{k=0}^{N-1} \eta_k^2 J_{\eta_k}(S,x) = o\,(N\ \zeta(N))\ \text{ a.u.} \tag{27}$$

We now indicate that by hypothesis η_k is a decreasing sequence and that $J_{\eta_k}(S,x)$ is increasing as η_k goes to zero. It follows then

$$\sum_{k=0}^{2N-1} \eta_k^2\ J_{\eta_k}(S,x) \geq \frac{(2N-1)}{2}\eta_{2N}^2 J_{\eta_N}(S,x). \tag{28}$$

Hence from (27)

$$\eta_{2N}^2\ J_{\eta_N}(S,x) = o\,(\zeta(2N))\ \text{ a.u.} \tag{29}$$

then (24) follows.

It is easy to obtain an explicit modulus of convergence by taking $a_n = \frac{C\ \phi(2^{n+2})}{\varphi(2^{n+1})} = \frac{16\ ||f||_2^2}{\zeta(2^{n+1})}$ with $q_n = (\sum_{r=n}^{\infty} a_r)^{-1/2}$ in Theorem 1 and backtraking the computations in that theorem to Lemma 3. □

Corollary 1. *Let q and $f(x)$ be as in the previous theorem. Then for any positive, increasing function λ defined on $(0,\infty)$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda(2^n)} < \infty, \tag{30}$$

we have the following asymptotic result in $\eta \searrow 0$:

$$J_\eta(S,x)\ \eta^2 = o\,(\lambda(1/\eta))\ \text{ a.u.} \tag{31}$$

Proof. To establish (31) it is sufficient to prove the asymptotics:

$$J_{2^{-N}}(S,x)\ 2^{-2N} = o\left(\lambda(2^{N-1})\right)\ \text{ a.u.} \tag{32}$$

In fact, given $0 < \eta \leq 2$ there exists an integer $N = N(\eta)$ such that $2^{-N} \leq \eta < 2^{-N+1}$ and then

$$\begin{aligned} J_\eta(S,x)\ \eta^2 &\leq J_{2^{-N}}(S,x)\ 2^{2(-N+1)} \\ &= 4\ J_{2^{-N}}(S,x)\ 2^{-2N}. \end{aligned}$$

And as $\lambda(2^{N-1}) \leq \lambda(1/\eta)$ we have that:

$$\frac{J_\eta(S,x)\ \eta^2}{\lambda(1/\eta)} \leq 4\ \frac{J_{2^{-N}}(S,x)\ 2^{-2N}}{\lambda(2^{N-1})}.$$

Observe that $\eta \searrow 0$ implies $N = N(\eta) \nearrow \infty$, thus (32) implies (31). If we take $N = 2^n$, $\eta_n = \frac{1}{n}$, and $\zeta(x) = \lambda(\frac{x}{4})$ and apply Theorem 2, we have that:

$$\frac{1}{4} J_{2^{-N}}(S,x)\ 2^{-2N} = o\left(\lambda(2^{N-1})\right)\ \text{ a.u.} \tag{33}$$

Then (31) follows. □

Corollary 2. *Let $f(x)$ be as in the previous theorem. Then for any $\epsilon > 0$ and integer $r > 0$ we have the following asymptotic result in $\eta \searrow 0$:*

$$J_\eta(S,x)\ \eta^2 = o\left((\log_1(1/\eta)) \ldots (\log_{r-1}(1/\eta))(\log_r(1/\eta))^{1+\epsilon}\right)\ \text{ a.u.} \tag{34}$$

Proof. This result follows from the convergence of the series:

$$\sum_{n=r}^{\infty} \frac{1}{\log_1(2^n)\dots\log_{r-1}(2^n)\ (\log_r(2^n))^{1+\epsilon}} \tag{35}$$

and the above theorem. □

As mentioned previously, the results for Chacon-Ornstein averages $S_n(x) = A_n f(x)$ specialize to the Cesaro averages by taking $q_n(x) = 1$ and assuming the extra condition $T1 \leq 1$.

Appendix A. Background Results

We need the following result about series:

Proposition 1. *Let a_n be a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} a_n < \infty$. Then there exists an unbounded nondecreasing sequence of positive real numbers q_n such that $\sum_{n=0}^{\infty} q_n a_n < \infty$.*

Proof. If only a finite number of the a_n are non zero the proposition is trivial. Now, we suppose that there are infinite nonzero a_n, in this case we take $q_n = (\sum_{r=n}^{\infty} a_r)^{-1/2}$, obviously q_n is an unbounded nondecreasing sequence of positive real numbers which satisfy

$$q_n a_n = \frac{q_{n+1}^2 - q_n^2}{q_{n+1}^2 q_n} \leq 2\left(\frac{1}{q_n} - \frac{1}{q_{n+1}}\right) \tag{36}$$

Since $\sum_{n=0}^{\infty}\left(\frac{1}{q_n} - \frac{1}{q_{n+1}}\right) = \frac{1}{q_0}$, we have that $\sum_{n=0}^{\infty} q_n a_n < \infty$. □

Lemma 3. *Given a sequence of functions $a_n \in L^1(X)$, $n = 0, 1, \dots$ with $a_n(x) \geq 0$ if*

$$\sum_{n=0}^{\infty} \int a_n(x)\ d\mu(x) < \infty \tag{37}$$

then

$$a_n(x) = o(1)\ a.u. \tag{38}$$

Proof. Let $a_n \geq \int a_n(x)\ d\mu(x)$ with $\sum_{n=0}^{\infty} a_n < \infty$ and q_n as in Proposition (1). Let $\delta_1 > 0$ and $N_1 = N(\delta_1)$ be such that

$$\sum_{n=N_1}^{\infty} q_n a_n \leq \delta_1$$

Define $B = \{x \in X :\ \exists\ n \geq N_1\ q_n a_n(x) \geq 1\}$. Let $B_n = \{x \in X : q_n a_n(x) \geq 1\}$, since $B \subseteq \bigcup_{n=N_1}^{\infty} B_n$ we have

$$\mu(B) \leq \sum_{n=N_1}^{\infty} \mu(B_n) \leq \sum_{n=N_1}^{\infty} \int_{B_n} q_n a_n(x)\ d\mu(x) \leq \sum_{n=N_1}^{\infty} q_n a_n \leq \delta_1.$$

Now given $\delta_2 > 0$, choose N_2, such that $N_2 \geq N_1$ and $q_n \geq 1/\delta_2\ \forall n \geq N_2$. Then for all $x \in B^c$ $a_n(x) \leq \delta_2$, $\forall n \geq N_2$. □

Remark 2. *It is clear that we can obtain $q_n a_n(x) = o(1)$ a.u. by iterating Lemma 3, however, this stronger statement will not improve the asymptotic given by (24).*

Appendix B. Upcrossing Inequalities

Here we describe Bishop's general results on upcrossing inequalities as presented in [2]. The general result represents a constructive outgrowth of the Chacon-Ornstein theorem. Let T be a linear operator on L^1 such that $T \geq 0$ and $||Tf||_1 \leq ||f||_1$ for all $f \in L^1$. Let $\{f_0, f_1, \dots, f_K\}$ denote a set of measurable functions such that $(f_i)_+ \in L^1$ and $T(\sum_{j\in\Omega} f_j)_+ \geq \sum_{j\in\Omega} f_{j+1}$ for any finite subset $\Omega \subset \{0, \dots, K-1\}$. Moreover, measurable functions $p_i(x) \geq 0, i = 0, \dots, K$ are given such that $Tp_i(x) \leq p_{i+1}(x)$. We now define an integer valued function $w_K(x)$:

$$w_K(x) = \sup\{k : \zeta = (u_r, v_r)_{r=1,\dots,k.}\} \tag{39}$$

where the sequence ζ satisfies,

$$-1 \leq u_1 < v_1 < u_2 < \dots < v_k \leq K \tag{40}$$

$$\sum_{j=0}^{u_r} f_j(x) \leq \sum_{j=0}^{v_r} (f_j(x) - p_j(x)) \quad r = 1, \dots, k \tag{41}$$

and

$$\sum_{j=0}^{u_{r+1}} f_j(x) \leq \sum_{j=0}^{v_r} (f_j(x) - p_j(x)) \quad r = 1, \dots, k-1 \tag{42}$$

To specialize these general definitions to count upcrossings for the Chacon-Ornstein averages $A_n f(x)$ take: $f_j(x) = T^j f(x) - \alpha\ q_j(x))$ and $p_j(x) = \eta\ q_j(x)$. To specialize the general definitions to count upcrossings in the setting of powers of T namely $P_n f(x)$ take: $f_j(x) = T^j f(x) - T^{j-1} f(x) - \alpha\ q_j(x)$ for $j \geq 1$, $f_0(x) = f(x)\ \alpha\ q_0(x)$ for $j = 0$ and $p_j(x) = \eta\ q_j(x)$. Each of this specializations will define a function $w_K(x)$, denote by $w_{\eta,K,\alpha}(S, x)$, that satisfies the property: $U_{\eta,K,\alpha}(S, x) \leq w_{\eta,K,\alpha}(S, x)$. For the case $f(x) \geq 0$, $\alpha \geq 0$, $\eta > 0$, Bishop's theorem ([2]) in both settings is:

Theorem 3.

$$\int p_0(x)\ w_{\eta,K,\alpha}(S, x)\ d\mu(x) \leq \int (f(x) - \alpha\ q_0(x))_+\ d\mu(x).$$

The key property $U_{\eta,K,\alpha} \leq w_{\eta,K,\alpha}$, is easily proven. Bishop also indicates ([2]) that further specializations of the general result imply that $w_K(x)$ becomes a majorization for the number of upcrossings in the context of Lebesgue's differentiation theorem and martingale convergence theorem. This indicates that analogous of our Theorem 2 are plausible in these other settings as well.

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Departamento de Matematicas. Facultad de Ciencias Exactas y Naturales. Universidad Nacional de Mar del Plata. Funes 3350, Mar del Plata 7600, Argentina.
E-mail address: pedrojc@mdp.edu.ar

Department of Mathematics, Physics and Computer Science, Ryerson Polytechnic University, Toronto, Ontario M5B 2K3, Canada.
E-mail address: ferrando@acs.ryerson.ca